

Q1.

Let $\varepsilon > 0$. Then

$$\begin{aligned} & P(\min\{X_1, \dots, X_n\} > \varepsilon) \\ &= P(\{X_1 > \varepsilon\} \cap \{X_2 > \varepsilon\} \cap \dots \cap \{X_n > \varepsilon\}) \\ &= P(\{X_1 > \varepsilon\}) \cdot P(\{X_2 > \varepsilon\}) \cdots P(\{X_n > \varepsilon\}) \quad (\text{independence}) \\ &= (1 - \varepsilon)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

As ε was arbitrary $\min\{X_1, \dots, X_n\} \xrightarrow{p} 0$.

Q2

Since $\{X_n\}$ and $\{Y_n\}$ are UI,

$$\forall \varepsilon > 0 \exists K > 0$$

$$\mathbb{E}(|X_n|; |X_n| > \frac{K}{2}) < \varepsilon/2$$

$$\text{and } \mathbb{E}(|Y_n|; |Y_n| > \frac{K}{2}) < \varepsilon/2.$$

a) Fix $\varepsilon > 0$, and let K be as above

$$\begin{aligned} & \text{Then, } \mathbb{E}(|X_n + Y_n|; |X_n + Y_n| > K) \\ & \leq \mathbb{E}(|X_n + Y_n|; \max\{|X_n|, |Y_n|\} > \frac{K}{2}) \\ & \leq \mathbb{E}(2 \max\{|X_n|, |Y_n|\}; \max\{|X_n|, |Y_n|\} > \frac{K}{2}) \\ & \leq 2 \max\left\{ \mathbb{E}(|X_n|; |X_n| > \frac{K}{2}), \mathbb{E}(|Y_n|; |Y_n| > \frac{K}{2}) \right\} \\ & \leq \varepsilon \quad \text{as required.} \end{aligned}$$

b) No. let $X_n = Y_n = Z$
 where $Z \in L^1$ but $Z \notin L^2$.

For example Z arising from pdf
 $f(x) = \begin{cases} 2x^{-3} & x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$ Then $\int f = 1$,

$$\mathbb{E}(Z; |Z| > K) = \int_K^\infty 2x^{-2} dx = \frac{2}{K} \rightarrow 0 \text{ as } K \rightarrow \infty.$$

$$\text{but } \mathbb{E}(|X_n Y_n|; |X_n Y_n| > K) = \int_K^\infty 2x^{-1} dx = \infty.$$

Q3 First assume $\mu_n, \sigma_n^2 \leq K$ for some $K > 0$. Then $\|X_n\|_2^2 = \mathbb{E}(|X_n|^2) = \text{Var}(X_n) + \mathbb{E}(X_n)^2 \leq K + K^2 \leq 2K^2 < \infty$.

Hence the family $\{X_n\}$ is uniformly bounded.

Now for the converse assume that μ_n or σ_n^2 is unbounded. If μ_n is unbounded $\exists n_k$ s.t.

$|\mu_{n_k}| \geq k \geq 1$ But then $\exists n_k$ s.t.

$$\mathbb{E}(|X_{n_k}|; |X_{n_k}| > k) \geq \frac{1}{2}k \geq \frac{1}{2}.$$

Hence $\{X_n\}$ is not U.I. So, it suffices

to check $\sigma_{n_k}^2 \rightarrow \infty$ but μ_n bounded.

Let $K = \frac{\sqrt{2\sigma_{n_k}^2}}{3}$ and assume WLOG that $|\mu_n| < K$ and $K \geq 1$ for all n .

$$\begin{aligned} \text{Then } \mathbb{E}(X_{n_k} ; |X_{n_k}| > K) &\geq \int_K^\infty \frac{y}{\sqrt{2\pi}\sigma_{n_k}} e^{-\frac{(y-\mu_{n_k})^2}{2\sigma_{n_k}^2}} dy \\ &\geq \frac{1}{\sqrt{2\pi}\sigma_{n_k}} \int_K^{2K} e^{-\frac{(y+K)^2}{2\sigma_{n_k}^2}} dy \quad 3K = \sqrt{2\sigma_{n_k}^2} \\ &\geq \frac{1}{\sqrt{2\pi}\sigma_{n_k}^2} K e^{-\frac{(3K)^2}{2\sigma_{n_k}^2}} = \frac{K}{\sqrt{\pi} 3K} e^{-1} = \frac{1}{3e\sqrt{\pi}} > 0. \end{aligned}$$

Hence $\{X_n\}$ is not UI.

[Q4 further below]

Q5 We first show that X_n is a martingale.

$$\mathbb{E}(X_n | \mathcal{F}_{n-1}) = \frac{1}{2} X_{n-1}^2 + \frac{1}{2} (2X_{n-1} - X_{n-1}^2) = X_{n-1}.$$

Hence also $\mathbb{E}(X_n) = \mathbb{E}(X_0) = a$.

Note that $x \mapsto x^2$, $x \mapsto 2x - x^2$ map $(0,1)$ into $(0,1)$

Hence $X_n \in (0,1)$ as $X_0 = a \in (0,1)$.

Further, X_n is uniformly bounded and thus

X_n is a UI martingale

and $X_n \rightarrow X_\infty$ a.s. with $E(X_n) = E(X_\infty) = a$.

Note that X_∞ cannot be in $(0, 1)$ a.s.

We will show this by contradiction: Let

Assume that $P(X_\infty \in (0, 1)) > 0$. Then,

there exists a small region $(x, x+\varepsilon)$ for $x > 0$ and $x+\varepsilon < 1$, with $P(X_\infty \in (x, x+\varepsilon)) > 0$.

But then $X_n \in (x, x+\varepsilon)$ for large enough n .

Hence $X_{n+1} = X_n^2$ or $2X_n - X_n^2$ and for small enough ε , only depending on x . But then $X_{n+1} \notin (x, x+\varepsilon)$ a contradiction. Hence $X_\infty \in \{0, 1\}$.

Alternatively, note that X_n and X_{n-1} converge to the same X_∞ . Hence

$$|X_{n+1} - X_n| \rightarrow 0. \quad \text{Since}$$
$$|X_{n+1} - X_n| = \begin{cases} |X_n^2 - X_n| & \text{with prob } \frac{1}{2} \\ |2X_n - X_n^2 - X_n| & \text{---} \end{cases} \stackrel{\text{a.s.}}{=} |X_n^2 - X_n|$$

We must have $|X_\infty^2 - X_\infty| = |X_\infty(X_\infty - 1)| = 0 \Rightarrow X_\infty \in \{0, 1\}$.

Now $E(X_\infty) = a$, which gives $P(X_\infty = 1) = a$ and

$$P(X_\infty = 0) = 1 - a. \quad \square$$

Q4 a) Clearly $E(Z_0) = 1$.

Inductively, if $E(Z_n) = m^n$, we have

$$\begin{aligned} E(Z_{n+1}) &= E(E(Z_{n+1} | \tilde{\mathcal{F}}_n)) \\ &= E\left(E\left(\sum_{i=1}^{Z_n} X_{i,n+1} \mid \tilde{\mathcal{F}}_n\right)\right) \\ &= E\left(\sum_{i=1}^{Z_n} E(X_{i,n+1} \mid \tilde{\mathcal{F}}_n)\right) \\ &= E\left(Z_n E(X)\right) = E(m \cdot Z_n) \\ &= m \cdot E(Z_n) = m \cdot m^n = m^{n+1}. \end{aligned}$$

This completes the inductive step.

b) Similar to above,

$$\begin{aligned} E(M_{n+1} \mid \tilde{\mathcal{F}}_n) &= \frac{1}{m^{n+1}} E(Z_{n+1} \mid \tilde{\mathcal{F}}_n) \\ &= \frac{1}{m^{n+1}} E\left(\sum_{i=1}^{Z_n} X_{i,n+1} \mid \tilde{\mathcal{F}}_n\right) = \frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} E(X_{i,n+1} \mid \tilde{\mathcal{F}}_n) \\ &= \frac{1}{m^{n+1}} Z_n \cdot m = \frac{Z_n}{m^n} = M_n \quad \text{as required.} \end{aligned}$$

Since $Z_n \geq 0$ and so $M_n \geq 0$, Doob's convergence

Theorem gives $M_n \rightarrow M_\infty$ a.s.

$$\begin{aligned}
c) \quad & \text{We compute } \mathbb{E}((M_{n+1} - M_n)^2) \\
&= \mathbb{E}\left(\left(\frac{1}{m^{n+1}} \sum_{i=1}^{Z_n} X_{i,n+1} - \frac{1}{m^n} Z_n\right)^2\right) \\
&= \frac{1}{m^{2(n+1)}} \mathbb{E}\left(\mathbb{E}\left(\left(\sum_{i=1}^{Z_n} X_{i,n+1} - m \cdot Z_n\right)^2 \mid \mathcal{F}_n\right)\right) \\
&= \frac{1}{m^{2(n+1)}} \mathbb{E}\left(\mathbb{E}\left(\underbrace{\left(\sum_{i=1}^{Z_n} (X_{i,n+1} - m)\right)^2}_{\text{sum of } Z_n \text{ many iid r.v. with mean 0 \& variance } \sigma^2} \mid \mathcal{F}_n\right)\right) \\
&= \frac{1}{m^{2(n+1)}} \mathbb{E}(Z_n \cdot \sigma^2) \\
&= \sigma^2 \frac{m^n}{m^{2n+2}} = \sigma^2 m^{-n-2}.
\end{aligned}$$

$$\text{Thus } \mathbb{E}(M_n^2) = \sum_{k=1}^n \mathbb{E}(M_k - M_{k-1})^2 = \sum_{k=1}^n \sigma^2 m^{-k-2} < \infty.$$

Hence $\mathbb{E}(M_n - M_\infty) \rightarrow 0$ and

$$\mathbb{E}(M_\infty) = \lim_n \mathbb{E}(M_n) = 1.$$

Since $M_n \geq 0$, we must have $P(M_\infty > 0) > 0$.

d) We have a.s. convergence, since

$M_n = Z_n$ is still a martingale (ident. proof as before).

Since $Z_n \geq 0$, we have $Z_n \rightarrow Z_\infty$ a.s.

Hence $Z_n, Z_{n-1} \rightarrow Z_\infty$ a.s. and

$$|Z_n - Z_{n-1}| \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

$$\begin{aligned} \text{Now } |Z_n - Z_{n-1}| &= \left| \sum_{i=1}^{Z_{n-1}} X_{i;n} - Z_{n-1} \right| \\ &= \left| \sum_{i=1}^{Z_{n-1}} (X_{i;n} - 1) \right|. \end{aligned}$$

Since $X_{i;n} - 1 = \begin{cases} 1 & \text{with prob } \frac{1}{2} \\ -1 & \text{with prob } \frac{1}{2} \end{cases}$, we must

have, for n large, $|Z_k - Z_{k-1}| < 1$
for all $k \geq n$. But then Z_k must be
even for all $k \geq n$ and $\frac{Z_k}{2}$ many $X_{i;k}$
must be $+1$ & $\frac{Z_k}{2}$ must be -1 for all $k \geq n$

This can only happen (with positive probability)
if $Z_k = 0$.

Alternatively, we want to prove that
 $P(Z_k = 0 \text{ for large enough } k) = 1$.

Write $q = P(Z_k = 0 \text{ for large enough } k)$.

Note that $q = P(Z_1 = 0) + P(Z_1 = 2 \text{ and the desc. of both subtrees are eventually extinct.})$

$$\text{So } q = \frac{1}{2} + \frac{1}{2} q^2 \Rightarrow q^2 - 2q + 1 = 0$$

$$\Rightarrow (q-1)^2 = 0. \text{ Hence } q = 1. \quad \square$$

Q6 We show $E(X_n) = 0$ from which $X_n = 0$ follows.

For a contradiction assume $E(X_n) \geq c$ for some $c > 0$ and $n \in \mathbb{N}$

Then by the submartingale property $E(X_k) \geq c > 0$

for all $k \geq n$. Let $\delta > 0$ be small

enough s.t. $E(X_n; \frac{c}{8\delta}) \leq \frac{c}{8}$ (using UI)

and let $k \geq n$ be large enough s.t.

$P(X_k \leq \frac{c}{8}) \geq 1 - \delta$ which exists as $X_n \xrightarrow{\text{a.s.}} 0 \Rightarrow X_k \xrightarrow{p} 0$.

Now
$$c \leq E(X_k) = E(X_k; X_k \leq \frac{c}{8}) + E(X_k; \frac{c}{8} \leq X_k \leq \frac{c}{\delta}) + E(X_k; X_k \geq \frac{c}{\delta})$$

$$= \frac{c}{8} + \delta \cdot \frac{c}{\delta \cdot 8} + \frac{c}{8}$$

$$= \frac{3}{8} c, \text{ a contradiction.}$$

Hence $E(X_n) = 0 \quad \forall n \in \mathbb{N}$ and as $X_n \geq 0$,

we must have $X_n = 0$ a.s.

Q7. At time n , the urn contains n balls. We have 4 cases:

Case	Probability
We pick red & add 2 red balls	$\frac{X_n}{n} \cdot p = a_1$
We pick green & add 2 green balls	$(1 - \frac{X_n}{n}) p = a_2$
We pick red & add 1 red, 1 green ball	$\frac{X_n}{n} \cdot q = a_3$
We pick green & add 1 red, 1 green ball	$(1 - \frac{X_n}{n}) q = a_4$

Hence, $E(X_{n+1} | \mathcal{F}_n)$

$$\begin{aligned}
 &= (X_n + 1)(a_1 + a_4) + X_n(a_2 + a_3) \\
 &= X_n(a_1 + a_2 + a_3 + a_4) + (a_1 + a_4) \\
 &= X_n + \frac{X_n}{n}p + (1 - \frac{X_n}{n})q \\
 &= X_n \left(1 + \frac{1-q}{n} - \frac{q}{n} \right) + q = X_n \left(\frac{n+1-q-q}{n} \right) + q \\
 &= X_n \left(\frac{n+1-2q}{n} \right) + q
 \end{aligned}$$

Hence,

$$E(Y_{n+1} | \mathcal{F}_n) = E \left(\binom{n+1-2q}{n}^{-1} \cdot \left(X_{n+1} - \frac{n+1}{2} \right) | \mathcal{F}_n \right)$$

$$\begin{aligned}
&= \frac{n! (n+1-2q-n)!}{(n+1-2q)!} \left(\mathbb{E}(X_{n+1} | \mathcal{F}_n) - \frac{n+1}{2} \right) \\
&= \frac{n! (1-2q)!}{(n+1-2q)!} \left(X_n \cdot \frac{n+1-2q}{n} + q - \frac{n+1}{2} \right) \\
&= \frac{n \cdot (n-1)! (1-2q)!}{(n+1-2q) \cdot (n-2q)!} \cdot \frac{n+1-2q}{n} \left(X_n + \frac{n}{n+1-2q} \left(q - \frac{n+1}{2} \right) \right) \\
&= \frac{(n-1)! (1-2q)!}{(n-2q)!} \left(X_n - \frac{\frac{1}{2}n^2 + \frac{1}{2}n - qn}{n+1-2q} \right) \\
&= \binom{n-2q}{n-1}^{-1} \left(X_n - \frac{n}{2} \right) = Y_n.
\end{aligned}$$

Hence Y_n is a martingale.

Q8 Lévy's upward theorem

a) By the tower property, $\mathbb{E}(M_n | \tilde{\mathcal{F}}_{n-1})$
 $= \mathbb{E}(\mathbb{E}(X | \mathcal{F}_n) | \tilde{\mathcal{F}}_{n-1}) = \mathbb{E}(X | \tilde{\mathcal{F}}_{n-1}) = M_{n-1}$.

b) Further, $(M_n) \subseteq \mathcal{G} = \{ \mathbb{E}(X | \mathcal{H}) : \mathcal{H} \in \mathcal{F} \text{ sub } \sigma\text{-alg.} \}$

We have shown that \mathcal{G} is UI and so $(M_n)_n$ is.

c) Let

$$\mu_1(F) = \mathbb{E}(Y; F), \quad \mu_2(F) = \mathbb{E}(M_\infty; F)$$

for $F \in \tilde{\mathcal{F}}_\infty$. Note that for all $F \in \tilde{\mathcal{F}}_\infty$ (and so $F \in \tilde{\mathcal{F}}_n$)

$$\begin{aligned} \mathbb{E}(Y; F) &= \mathbb{E}(\mathbb{E}(X | \tilde{\mathcal{F}}_\infty); F) \\ &= \int_F \mathbb{E}(X | \tilde{\mathcal{F}}_\infty) dP = \mathbb{E}(X; F) \end{aligned}$$

For $E \in \mathcal{F}_n$, we also have

$$\mathbb{E}(M_\infty | \mathcal{F}_n) = X_n \quad \text{and so} \quad \mathbb{E}(M_\infty; E) = \mathbb{E}(X; E).$$

Now $E \in \bigcup \tilde{\mathcal{F}}_n$ generate a π -system and so

$$\mathbb{E}(M_\infty; E) = \mathbb{E}(Y; E) \quad \forall E \in \bigcup \tilde{\mathcal{F}}_n$$

$$\Rightarrow \mu_1(F) = \mu_2(F) \quad \forall F \in \sigma(\bigcup \tilde{\mathcal{F}}_n).$$

d) Finally, since M_∞ and Y are \mathcal{F}_∞ -measurable,
 $F = \{M_\infty - Y > 0\} \in \mathcal{F}_\infty$ and $\mu_1(F) = \mu_2(F)$.

$$\text{Hence } \int_F M_\infty dP = \int_F Y dP \Rightarrow \int_F M_\infty - Y dP = 0.$$

Hence $M_\infty = Y$ a.s., which proves the theorem \square

Q8 Kolmogorov 0-1 law

Consider I_F for $F \in \mathcal{T}$. This
is clearly bounded and by Levy's Upward Th^m,

$$I_F = \mathbb{E}(I_F | \mathcal{F}_\infty) = \lim \mathbb{E}(I_F | \tilde{\mathcal{F}}_n).$$

I_F is \mathcal{T}_n measurable $\forall n$ and thus independent
of $\tilde{\mathcal{F}}_n$.

$$\text{So, } I_F = \lim \mathbb{E}(I_F | \tilde{\mathcal{F}}_n) = \lim \mathbb{E}(I_F) = P(F).$$

and can only take values in $\{0, 1\}$