Problem Set 4

PhD Course in Probability and Statistics, Part I

Below, you find a number of exercises you can attempt during the course. They are not assessed but complement the material in class. It is strongly recommended you try them without looking at the solutions (which will be posted a little bit later).

Problems

1. Let $X_1, X_2, \ldots$ be a sequence of independent uniformly distributed random variables on the interval $[0,1]$. Prove (directly from definition) that $\min(X_1, X_2, \ldots, X_n) \to_p 0$ as $n \to \infty$.

2. Let $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ be uniformly integrable sequences of random variables.
   (a) Prove that the sequence $\{X_n + Y_n \mid n \geq 1\}$ is also uniformly integrable.
   (b) Is $\{X_n Y_n \mid n \geq 1\}$ also uniformly integrable?

3. For $n \in \mathbb{N}$, let X_n be normally distributed with mean μ_n and variance σ_n^2. Prove that the family $\{X_n \mid n \geq 1\}$ is uniformly integrable if and only if both μ_n and σ_n^2 are uniformly bounded.

4. Let $X : \Omega \to \{0, 1, 2, \ldots\}$ be a random variable with mean $m = \mathbb{E}(X) > 1$ and variance $\sigma^2 = \mathrm{Var}(X) < \infty$. We define the *Galton-Watson* process Z_n associated with X by,

$$Z_0 = 1 \quad \text{and} \quad Z_n = \sum_{j=1}^{Z_{n-1}} X_{j,n} \quad \text{for } n \geq 1,$$

   where $X_{j,n}$ are independent random variables with the same distribution as X.
   (a) Show that $\mathbb{E}(Z_n) = m^n$.
   (b) Prove that $M_n = m^{-n} Z_n$ is a martingale and that it converges to some random variable M_∞ almost surely.
   (c) Show that $\mathbb{E}(M_n) \to \mathbb{E}(M_\infty) = 1$ (Hint show that the martingale is in L^2). Conclude that $\mathbb{P}(M_\infty \neq 0) > 0$.
   (d) Now let $X = 0$ with probability $\frac{1}{2}$ and $X = 2$ with probability $\frac{1}{2}$. Now $m = \mathbb{E}(X) = 1$. What can we say about M_∞?

5. Consider the following sequence of random variables: $X_0 = a$ for some $a \in (0,1)$, and

$$X_n = \begin{cases} X_{n-1}^2 & \text{with probability } \frac{1}{2}, \\ 2X_{n-1} - X_{n-1}^2 & \text{with probability } \frac{1}{2}, \end{cases}$$

   for $n > 0$. Prove that the sequence $X_0, X_1, \ldots$ converges almost surely. What are the possible limits? For each of the possible limits L, determine

$$\mathbb{P}(\lim_{n \to \infty} X_n = L).$$

6. Prove that if X_n is a non-negative, uniformly integrable submartingale for which $X_n \to 0$ holds almost surely, as $n \to \infty$, then $X_n = 0$ (a.s.) for all $n \in \mathbb{N}$.

7. Let $p \in (0,1)$ be fixed. We have an inexhaustible supply of red and green balls. In a bucket, there is initially one red ball. In each time step, we take a random ball from the bucket. With probability p, we replace it along with another ball of the same colour. With probability $q = 1-p$ we replace it and add a ball of the other colour. Let X_n be the number of red balls in the n-th step. Prove that

$$Y_n = (X_n - n/2) \cdot \binom{n-2q}{n-1}^{-1}$$

is a martingale.

8. In this exercise we will prove *Lévy's Upward Theorem* and give an alternative proof to Kolmogorov's $0-1$ law.
Theorem. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_n$ be a filtration. Define $M_n = \mathbb{E}(X) \mid \mathcal{F}_n)$. Then M_n is a martingale and

$$M_n \to Y := \mathbb{E}(X \mid \mathcal{F}_\infty),$$

almost surely and in L^1, where $\mathcal{F}_\infty = \sigma\left(\bigcup \mathcal{F}_n\right)$.

(a) Show that M_n is a martingale.
(b) Show that M_n is UI.
(c) Define measure μ_1, μ_2 on $(\Omega, \mathcal{F}_\infty)$ by

$$\mu_1(F) = \mathbb{E}(Y; F) \quad \text{and} \quad \mu_2(F) = \mathbb{E}(M_\infty; F).$$

Show that $\mu_1 = \mu_2$.
(d) Show that, almost surely, $Y = M_\infty$. (Hint: consider the expectation of the difference)

Recall Kolmogorov's $0-1$ law:
Theorem. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Define

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \ldots) \quad \text{and} \quad \mathcal{T} = \bigcap_n \mathcal{T}_n.$$

Then, for all $E \in \mathcal{T}$, we have $\mathbb{P}(E)$ is either 0 or 1.

(a) Use Lévy's Upward Theorem with $Y = I_E$ and show that

$$X = \mathbb{E}(X \mid \mathcal{F}_\infty) = \lim_n \mathbb{E}(X \mid \mathcal{F}_n).$$

(b) Show that $Y = \mathbb{P}(E)$ and prove the theorem.
(Hint: Use independence of $\mathcal{F}_n$ and $\mathcal{T}_n$)