Problem Set 3

PhD Course in Probability and Statistics, Part I

Below, you find a number of exercises you can attempt during the course. They are not assessed but complement the material in class. It is strongly recommended you try them without looking at the solutions (which will be posted a little bit later).

Problems

1. Suppose that X is a random variable that has moments of all orders, i.e., $\mathbb{E}(|X|^p) < \infty$ for all $p > 0$. Prove that

$$\lim_{p\to\infty} \left(\mathbb{E}(|X|^p)\right)^{1/p} = \inf\{K \geq 0 : \mathbb{P}(|X| > K) = 0\}.$$

(If the set $\{K \geq 0 : \mathbb{P}(|X| > K) = 0\}$ is empty, the infimum is ∞).

2. Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(X^2) < \infty$. We define the conditional variance with respect to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ by

$$\mathrm{Var}(X|\mathcal{G}) = \mathbb{E}((X - \mathbb{E}(X|\mathcal{G}))^2|\mathcal{G}).$$

Prove that

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\mathcal{G})) + \mathrm{Var}(\mathbb{E}(X|\mathcal{G})).$$

3. Let $Y_1, Y_2, \ldots$ be independent random variables with $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1-p$ ($p \in (0,1)$, $p \neq \frac{1}{2}$) for all i, and consider the simple biased random walk $X_n = \sum_{i=1}^n Y_i$.
   (a) Find a constant $\theta \neq 1$ such that θ^{X_n} is a martingale.
   (b) Find a (deterministic) function $f(n)$ such that $X_n - f(n)$ is a martingale.
   (c) Let a and b be positive integers. Determine the probability that X_n reaches the value a before the value $-b$.
   (d) Determine the expected number of steps until one of these two values is reached.

4. Prove: a previsible martingale X_n is almost surely constant, i.e., $X_n = X_0$ holds almost surely for all n.

5. Suppose that X and Y are integrable random variables such that $\mathbb{E}(X|\mathcal{G}) = Y$ and $\mathbb{E}(X^2|\mathcal{G}) = Y^2$. Prove that $X = Y$ almost surely.
   Hint: Consider $\mathbb{E}((X-Y)^2|\mathcal{G})$.

6. Let (X, Y) be a uniformly random point in the unit disk (centre at $(0,0)$, radius 1). Determine $\mathbb{E}(X \mid Y)$, $\mathbb{E}(|X| \mid Y)$, $\mathbb{E}(X \mid |Y|)$, and $\mathbb{E}(|X| \mid |Y|)$.

7. Let $Y_1, Y_2, \ldots$ be independent random variables that follow a normal distribution with mean 0 and variance 1. Set $S_n = Y_1 + Y_2 + \cdots + Y_n$. Prove that

(a) $X_n = e^{S_n - n/2}$ is a martingale.

(b) $X_n \to 0$ almost surely as $n \to \infty$.

(c) X_n^r is a supermartingale for $0 < r < 1$, and a submartingale for $r > 1$.