Problem Set 1

PhD Course in Probability and Statistics, Part I

Below, you find a number of exercises you can attempt during the course. They are not assessed but complement the material in class. It is strongly recommended you try them without looking at the solutions (which will be posted a little bit later).

Problems

1. Show that the two operations Δ (symmetric difference), defined by $A\Delta B = (A \setminus B) \cup (B \setminus A)$, and $\cap$ (intersection) satisfy the following properties:
   (a) Δ and $\cap$ are both commutative and associative.
   (b) They satisfy the distributive law: $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$.

2. Which of the following statements are true for all possible sequences A_n, B_n of sets?
   (a) $\limsup_{n\to\infty}(A_n \cap B_n) = (\limsup_{n\to\infty} A_n) \cap (\limsup_{n\to\infty} B_n)$
   (b) $\limsup_{n\to\infty}(A_n \cup B_n) = (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n)$
   (c) $\liminf_{n\to\infty}(A_n \cap B_n) = (\liminf_{n\to\infty} A_n) \cap (\liminf_{n\to\infty} B_n)$
   (d) $\liminf_{n\to\infty}(A_n \cup B_n) = (\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n)$

3. Prove: if $f : S \to \mathbb{R}$ is a measurable function on some measure space S with σ-algebra Σ, then so is $|f|$. Show by means of a counterexample that the converse is not necessarily true.

4. Let $\{A_n, n \geq 1\}$ be a sequence of events in a probability space.
   (a) Suppose that $\lim_{n\to\infty} P(A_n) = 1$. Prove that there exists an increasing subsequence $\{n_k, k \geq 1\}$ such that
   $$P\left(\bigcap_{k\geq 1} A_{n_k}\right) > 0.$$
   (b) Give an example of a sequence of events (in a probability space of your choice) with $P(A_n) \geq \frac{1}{2}$ for all $n \geq 1$ for which there is no such subsequence.

5. Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Prove the following inequalities:
   (a) $P\left(\bigcup_{k=1}^n A_k\right) \geq \sum_{k=1}^n P(A_k) - \sum_{1\leq j<k\leq n} P(A_j \cap A_k)$.
   (b) $P\left(\bigcup_{k=1}^n A_k\right) \leq \sum_{k=1}^n P(A_k) - \sum_{1\leq j<k\leq n} P(A_j \cap A_k) + \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k)$.

6. Let X be a random variable. Show: for every $\epsilon > 0$, there exists a bounded random variable X_ϵ (i.e., there exists a constant M such that $|X_\epsilon| \leq M$ holds almost surely) such that $P(X \neq X_\epsilon) < \epsilon$.

7. Suppose that X is an integer-valued random variable, and let m be a positive integer. Prove that

$$\sum_{n=-\infty}^{\infty} P(n < X \leq n+m) = m.$$

8. Prove: if $\{X_n, n \geq 1\}$ is a sequence of independent random variables, then the following two statements are equivalent:

   - $P\left(\sup_{n\geq 1} X_n < \infty\right) = 1$,
   - there exists an $a > 0$ such that $\sum_{n=1}^{\infty} P(X_n > a) < \infty$.

9. Let $\{A_n, n \geq 1\}$ be a sequence of independent events in a probability space, and suppose that $P(A_n) < 1$ for all n. Prove that the following two statements are equivalent:

   - $P(A_n$ occurs for at least one $n) = 1$,
   - $P(A_n$ occurs for infinitely many $n) = 1$.

   Why is $P(A_n) = 1$ forbidden?

10. Let $X_1, X_2, \ldots$ be independent random variables, where X_n follows a uniform distribution on the interval $\left[0, \frac{1}{n}\right]$ (equivalently, $X_n = \frac{Y_n}{n}$, where Y_n follows a uniform distribution on $[0, 1]$). Prove that $X = \sup_n X_n$ has the distribution function

$$F(x) = \lfloor \frac{1}{x} \rfloor! \cdot x^{\lfloor \frac{1}{x} \rfloor}, \quad 0 < x \leq 1.$$

    (For $x \leq 0$, $F(x) = 0$, and for $x > 1$, $F(x) = 1$.) Here, $\lfloor a \rfloor$ is the greatest integer $\leq a$.