Lecture 9:

Recall: that the conditional expectation of X conditioned on $\mathcal{G}$ (sub σ-algebra), $Y = \mathbb{E}(X \mid \mathcal{G})$ is the unique (a.s.) random variable s.t. Y is $\mathcal{G}$-measurable and

$$\int_G Y(\omega)\, d\mathbb{P} = \int_G X \, d\mathbb{P} \quad \forall\, G \in \mathcal{G}.$$

We saw that MCT, DCT, Fatou also hold for conditional expectation.

We also get a corresponding analogue of Jensen's inequality:

Th$^{\underline{m}}$ Let $g: I \to \mathbb{R}$ be a convex function on an interval $I \subseteq \mathbb{R}$. Assume $X: \Omega \to I$ and X and $g(X)$ are integrable. Then,

$$\mathbb{E}(g(X) \mid \mathcal{G}) \geq g(\mathbb{E}(X \mid \mathcal{G})) \quad \text{a.s.}$$

Simplification rules:

1) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H})$

for sub σ-algebras $\mathcal{G}, \mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{G}$.

2) $\mathbb{E}(Z \cdot X \mid \mathcal{G}) = Z \cdot \mathbb{E}(X \mid \mathcal{G})$

if Z is $\mathcal{G}$-measurable (completely determined by $\mathcal{G}$)

3) $\mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X \mid \mathcal{G})$

if $\mathcal{H}$ is independent of $X, \mathcal{G}$.

Special case: $\mathcal{G} = \{\emptyset, \Omega\}$.

If X is independent of $\mathcal{H}$, then

$\mathbb{E}(X \mid \mathcal{H}) = \mathbb{E}(X)$.

These can be proved by verifying conditions of the conditional expectation.

Product Spaces & Measures

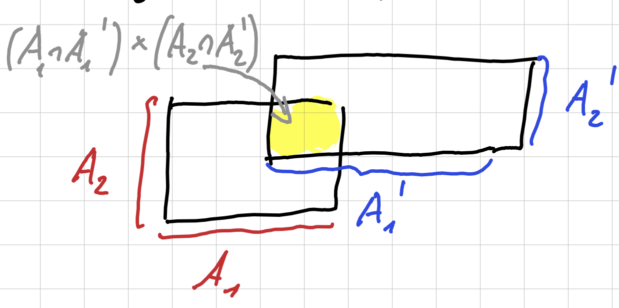
Given two measure spaces (S_1, Σ_1, μ_1) and (S_2, Σ_2, μ_2), we want to define their product as a measure space on $S = S_1 \times S_2$.

<u>Product σ-algebra</u>

$$\Sigma = \Sigma_1 \times \Sigma_2 \qquad \text{(notation)}$$

$$= \sigma\left(\bigcup_{A \in \Sigma_1} (A \times S_2) \cup \bigcup_{B \in \Sigma_2} (S_1 \times B) \right) \qquad \text{(definition)}$$

Remark: $\{A_1 \times A_2 : A_1 \in \Sigma_1, A_2 \in \Sigma_2\}$ is a π-system that generates Σ.



If f is a bounded measurable function on (S, Σ), then the projections

$$S_1 \to \mathbb{R} \qquad s_1 \mapsto f(s_1, s_2) \quad (s_2 \text{ fixed})$$

$$S_2 \to \mathbb{R} \qquad s_2 \mapsto f(s_1, s_2) \quad (s_1 \text{ fixed})$$

are measurable for all s_2, s_1, respectively.

Proof: This clearly holds for indicator functions of form

$$I_{A_1 \times A_2}(s_1, s_2) = \begin{cases} 1 & s_1 \in A_1 \ \ s_2 \in A_2 \\ 0 & \end{cases}$$

For arbitrary f use approximation by step functions. □

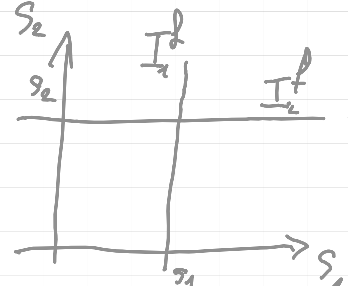
Product measure

Assume that μ_1, μ_2 are finite measures.

We can define the two functions

$$I_1^f(s_1) = \int_{S_2} f(s_1, s_2)\, d\mu_2$$

$$I_2^f(s_2) = \int_{S_1} f(s_1, s_2)\, d\mu_1$$



Lemma: For bounded measurable f, both of these are bounded & measurable.

Proof: For indicators $I_{A_1 \times A_2} = f$:

$$I_1^f(s_1) = \int_{S_2} I_{A_1 \times A_2}(s_1, s_2)\, d\mu_2 = \int_{S_2} I_{A_1}(s_1) I_{A_2}(s_2)\, d\mu_2$$

$$= I_{A_1}(s_1) \cdot \int_{S_2} I_{A_2}(s_2)\, d\mu_2 = I_{A_1}(s_1) \cdot \mu_2(A_2)$$

(analogous for $I_2^f(s_2)$)

For arbitrary f, we approximate by step functions. $\square$

Now for $F \in \Sigma$ take $f = I_F$ and define $\mu(F) = \int_{S_1} I_1^f\, d\mu_1 = \int_{S_1} \left(\int_{S_2} f(s_1, s_2)\, d\mu_2 \right) d\mu_1$

$$\overset{(*)}{=} \int_{S_2} I_2^f\, d\mu_2 = \int_{S_2} \left(\int_{S_1} f(s_1, s_2)\, d\mu_1 \right) d\mu_2$$

Th$^{\underline{m}}$ (Fubini's theorem): This is well-defined (i.e. (*) holds). In fact,

$$\int_{S_1} \int_{S_2} f\, d\mu_2\, d\mu_1 = \int_{S_2} \int_{S_1} f\, d\mu_1\, d\mu_2 = \int_{S_1 \times S_2} f\, d\mu$$

for all non-negative (or even integrable) f.

Here μ is the unique measure that satisfies $\mu(A_1 \times A_2) = \mu(A_1)\mu(A_2)$ $\forall A_1 \in \Sigma_1, A_2 \in \Sigma_2$.

<u>Proof:</u> When $f = I_{A_1 \times A_2}$,

$$\int_{S_1}\int_{S_2} I_{A_1 \times A_2}(s_1, s_2)\, d\mu_2\, d\mu_1 = \int_{S_1} I_{A_1}(s_1) \int_{S_2} I_{A_2}(s_2)\, d\mu_2\, d\mu_1$$

$$= \mu_1(A_1) \cdot \mu_2(A_2) = \int_{S_2}\int_{S_1} I_{A_1 \times A_2}(s_1, s_2)\, d\mu_1\, d\mu_2$$

For general f, we approximate by step functions.
Uniqueness follows since $\{A_1 \times A_2 : A_1 \in \Sigma_1, A_2 \in \Sigma_2\}$ is a π-system and so μ is determined uniquely by the values of $\mu(A_1 \times A_2)$. $\square$

This construction defines the product measure μ, also written as $\mu = \mu_1 \times \mu_2$.

<u>Remark:</u> We can extend this to products of several measure spaces/measures $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n$ or even countably infinite products $\mu = \mu_1 \times \mu_2 \times \ldots$

Example: The Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is the same as $\underbrace{\mathcal{L}^1 \times \mathcal{L}^1 \times \dots \times \mathcal{L}^1}_{n \text{ times}}$.

Remark: Fubini's theorem remains true for σ-finite measures but not necessarily otherwise!

Example: μ_1 Lebesgue measure on $[0,1]$

(not σ-finite) $\rightarrow$ μ_2 Counting measure on $[0,1]$

Let $f(s_1, s_2) = \begin{cases} 1 & s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{S_1} \int_{S_2} f \, d\mu_2 \, d\mu_1 = \int_{S_1} 1 \, d\mu_1 = 1$$

$$\neq$$

$$\int_{S_2} \int_{S_1} f \, d\mu_1 \, d\mu_2 = \int 0 \, d\mu_2 = 0$$

An application: formula for $\mathbb{E}(X)$

Suppose X is a non-negative r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$

Then,

$$\int_0^\infty \underbrace{\int_\Omega I(X \geq x)\, d\mathbb{P}}_{= \mathbb{P}(X \geq x)} dx = \int_\Omega \underbrace{\int_0^\infty I(X \geq x)\, dx}_{\int_0^X 1\, dx = X} d\mathbb{P}$$

$$\int_0^\infty \mathbb{P}(X \geq x)\, dx = \int X\, d\mathbb{P} = \mathbb{E}(X)$$

Conditional expectations (cont.)

Proof (that $\mathbb{E}(X \mid \mathcal{G})$ is a.s. unique):

Assume that there are two random variables Y, Y' that satisfy conditions and that $\mathbb{P}(Y = Y') \neq 1$. Then $\mathbb{P}(Y > Y') > 0$ or $\mathbb{P}(Y' < Y) > 0$. Assume without loss of generality it's the former

Note: $\{Y > Y'\} = \bigcup_{n \in \mathbb{N}} \{Y \geq Y' + \frac{1}{n}\}$

and for some n we have $\mathbb{P}(Y \geq Y' + \frac{1}{n}) > 0$.

Y, Y' are $\mathcal{G}$-measurable $\Rightarrow$ $Y - Y'$ is $\mathcal{G}$ measurable.

$\Rightarrow \{Y \geq Y' + \frac{1}{n}\} = \{Y - Y' \geq \frac{1}{n}\} \in \mathcal{G}$.

By condition (3), $\int_G Y \, d\mathbb{P} = \int_G X \, d\mathbb{P} = \int_G Y' \, d\mathbb{P}$

and $\int_{\{Y-Y' \geq \frac{1}{n}\}} Y \, d\mathbb{P} = \int_{\{Y-Y' \geq \frac{1}{n}\}} Y' \, d\mathbb{P}$

$$\Rightarrow \underbrace{\int_{\{Y-Y' \geq \frac{1}{n}\}} Y - Y' \, d\mathbb{P}}_{\geq \frac{1}{n} \mathbb{P}(Y-Y' \geq \frac{1}{n}) > 0} \overset{\text{a contradiction } ↯}{\longleftrightarrow} = 0$$

$\square$

We consider special case where X, Y have common density to find $\mathbb{E}(X|Y)$:

$$\mathbb{P}((X,Y) \in A) = \iint_A f_{X,Y}(x,y) \, dx \, dy .$$

We define the conditional density

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & \text{if } f_Y(y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y) \, dx$.

Note that for fixed y, $\int_{\mathbb{R}} f_{X|Y}(x|y) \, dx = \int \frac{f_{X,Y}(x,y)}{f_Y(y)} \, dx$

$= \frac{1}{f_Y(y)} \int f_{X,Y}(x,y) \, dx = 1.$

So $f_{X|Y}(x|y)$ is a density, provided that $f_Y(y) \neq 0$ ("the density of X given $Y=y$")

Now set $g(y) = \int_{\mathbb{R}} x \, f_{X|Y}(x|y) \, dx$

("the expected value of X given $Y=y$").