Lecture 4

Independence.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Def^n: Let $E_1, E_2, \ldots$ be events (finite or countable). We say that they are independent if $\mathbb{P}(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = \mathbb{P}(E_{i_1}) \cdot \mathbb{P}(E_{i_2}) \ldots \mathbb{P}(E_{i_k})$ for all choices of $i_1 < i_2 < \ldots < i_k$.

Example: Consider throw of a die:

E_1 = number is $\leq 2 = \{1, 2\}$, $\mathbb{P}(E_1) = 1/3$;

E_2 = number is even $= \{2, 4, 6\}$, $\mathbb{P}(E_2) = 1/2$.

Since $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(\{2\}) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = \mathbb{P}(E_1)\mathbb{P}(E_2)$ the events E_1, E_2 are independent.

Let E_3 = number is $\leq 3 = \{1, 2, 3\}$.

Then $\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(\{2\}) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(E_2)\mathbb{P}(E_3)$

So E_2 and E_3 are not independent.

Def$\underline{^n}$ Random variables $X_1, X_2, \ldots$ are said to be independent if for any choice of $i_1 < i_2 < \ldots < i_k$ and Borel sets $A_1, \ldots, A_k$, the events $\{X_{i_j} \in A_j\}$ $(j = 1, 2, \ldots, k)$ are independent.

That is, $\mathbb{P}(\{X_{i_1} \in A_1\} \cap \{X_{i_2} \in A_2\} \cap \ldots \cap \{X_{i_k} \in A_k\})$

$$= \prod_{j=1}^{k} \mathbb{P}(\{X_{i_j} \in A_j\})$$

Def$\underline{^n}$ Sub σ-algebras $\mathcal{G}_1, \mathcal{G}_2, \ldots$ are said to be independent if

$$\mathbb{P}(G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_k}) = \prod_{j=1}^{k} \mathbb{P}(G_{i_j})$$

for all choices of indices $i_1 < i_2 < \ldots < i_k$ and events $G_{i_j} \in \mathcal{G}_{i_j}$

<u>Remark</u>: Independence of events and random variables are special cases of independence of σ-algebras.

1) Let $E_1, E_2, \ldots$ be events.

Set $\mathcal{G}_j = \{\emptyset, E_j, E_j^c, \Omega\} = \sigma(E_j)$. Then,

$E_1, E_2, \ldots$ independent $\iff$ $\mathcal{G}_1, \mathcal{G}_2, \ldots$ independent.

2) Let $X_1, X_2, \ldots$ be random variables.
Let $\mathcal{G}_j = \sigma(X_j)$. Then,
$X_1, X_2, \ldots$ independent $\iff$ $\mathcal{G}_1, \mathcal{G}_2, \ldots$ independent.

<u>Lemma</u> Let $\mathcal{G}, \mathcal{H}$ be sub σ-algebras of $\mathcal{F}$.
and let $\mathcal{I}, \mathcal{J}$ be π-systems that generate $\mathcal{G}$ and $\mathcal{H}$: $\sigma(\mathcal{I}) = \mathcal{G}$ and $\sigma(\mathcal{J}) = \mathcal{H}$.
Then $\mathcal{G}, \mathcal{H}$ are independent if and only if
$$\mathbb{P}(I \cap J) = \mathbb{P}(I)\,\mathbb{P}(J) \text{ for all } I \in \mathcal{I},\ J \in \mathcal{J}. \quad (*)$$

<u>Proof</u>: "$\Rightarrow$" is clear as $\mathcal{I} \subseteq \mathcal{G}$, $\mathcal{J} \subseteq \mathcal{H}$.

"$\Leftarrow$": Assume $(*)$ holds. Define
$$\mathbb{P}_I(H) = \mathbb{P}(I \cap H) \quad \text{for all } H \in \mathcal{H}.$$
This is a measure on $(\Omega, \mathcal{H})$:

- $\mathbb{P}_I(\emptyset) = \mathbb{P}(I \cap \emptyset) = \mathbb{P}(\emptyset) = 0$
- $$\mathbb{P}_I\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(I \cap \bigcup_{i=1}^{\infty} A_i\right)$$
$$= \mathbb{P}\left(\bigcup_{i=1}^{\infty} I \cap A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(I \cap A_i)$$
$$= \sum_{i=1}^{\infty} \mathbb{P}_I(A_i) \quad \text{for disjoint } A_i.$$

Moreover, $P_I'(H) = \mathbb{P}(I) \cdot \mathbb{P}(H)$ is a measure on $(\Omega, \mathcal{H})$.

Since measures are uniquely determined by π-systems, $P_I'(H) = P_I(H)$ for <u>all</u> $H \in \mathcal{H}$ by using $(*)$ over all $H \in \mathcal{J}$.

So $\mathbb{P}(I \cap H) = \mathbb{P}(I)\mathbb{P}(H)$ for all $I \in \mathcal{I}$, $\underline{H \in \mathcal{H}}$

Repeating the argument on the left gives

$$\mathbb{P}(G \cap H) = \mathbb{P}(G)\mathbb{P}(H) \text{ for all } G \in \mathcal{G}, H \in \mathcal{H}. \quad \blacksquare$$

<u>Remark</u>
- $\{E_i\}$ is a π system.
- Events of the form $\{X_i \leq x\}$ form a π-system generating $\sigma(X_i)$. So, to verify that $X_1, X_2, ..$ are independent it suffices to check

$$\mathbb{P}(X_{i_1} \leq x_1 \ \& \ X_{i_2} \leq x_2 \ \& \ ..) = \mathbb{P}(X_{i_1} \leq x_1)\mathbb{P}(X_{i_2} \leq x_2)..$$

for all $i_1, i_2, ..$, and $x_1, x_2, ..$

Second Borel-Cantelli lemma:

Assume that $E_1, E_2, \dots$ are independent events and $\sum_{i=1}^{\infty} \mathbb{P}(E_i) = \infty$. Then,

$$\mathbb{P}(\limsup_{n\to\infty} E_n) = \mathbb{P}(E_n \text{ occurs infinitely often}) = 1$$

Proof: Recall that $\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} E_n$.

Its complement is $\bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} E_n^c$.

Now
$$\mathbb{P}(\bigcap_{m\geq n} E_m^c) \leq \mathbb{P}(\bigcap_{m=n}^{M} E_m^c) = \prod_{m=n}^{M} \mathbb{P}(E_m^c)$$

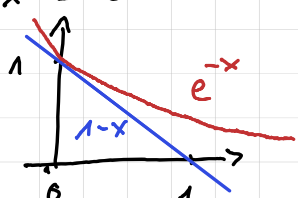
$$= \prod_{m=n}^{M} (1 - \mathbb{P}(E_m))$$

$$\leq \prod_{m=n}^{M} e^{-\mathbb{P}(E_m)}$$

$$= \exp\left(-\sum_{m=n}^{M} \mathbb{P}(E_m)\right)$$

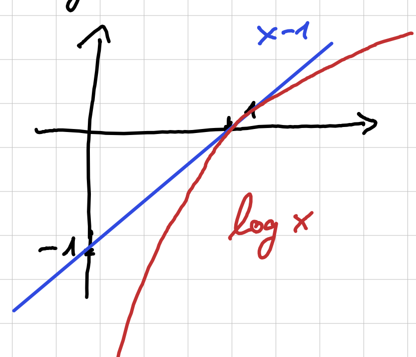
$$\to \text{"}\exp(-\infty)\text{"} = 0.$$

$1 - x \leq e^{-x}$



$\Updownarrow$

$\log x \leq x - 1$



Hence $\mathbb{P}(\bigcap_{m\geq n} E_m^c) = 0 \quad \forall n$

and by countable unions,

$\mathbb{P}(\bigcup \bigcap E_m^c) = 0. \qquad \square$

Remark: Independence is crucial!
If e.g. $E_1 = E_2 = \dots$, then $\mathbb{P}(\limsup E_n) = \mathbb{P}(E_n)$ which can take any value in $[0,1]$.

Example: Take a random card from a deck of n cards on the n-th draw. Assume the cards are labelled $\{1, 2, \dots, n\}$. Let
$E_n = \{$ card 1 is drawn on n-th draw $\}$.
Assuming all draws are independent and uniform in probability, $\mathbb{P}(E_n) = 1/n$. Hence, by the second BC-lemma, $\mathbb{P}($"1 is drawn infinitely often"$)$
$= \mathbb{P}(\limsup_n E_n) = 1$, since $\sum \mathbb{P}(E_n) = \sum \frac{1}{n} = \infty$.

<u>Example</u>: "Monkey & typewriter"
A monkey types a sequence of random characters on a keyboard. Assume each character has pos. prob. of occurring (at least $\varepsilon > 0$). Let S be a fixed string of length s. Then,

$$\begin{cases} \mathbb{P}(\text{First } s \text{ characters are exactly } S) \geq \varepsilon^s \\ \mathbb{P}(\text{Characters } s+1, \dots, 2s \text{ are exactly } S) \geq \varepsilon^s \\ \vdots \end{cases}$$

independent! $\displaystyle\sum_{k=1}^{\infty} \mathbb{P}(k\text{-th set of } s \text{ characters is exactly } S)$

$$\geq \sum_{k=1}^{\infty} \varepsilon^s = \infty.$$

By the second BC. lemma:

$\mathbb{P}(\text{monkey types } S \text{ infinitely often}) = 1.$

Note: If $E_1, E_2, \dots$ is a sequence of independent events, and since

$$\sum \mathbb{P}(E_j) = \infty \quad \text{or} \quad \sum \mathbb{P}(E_j) < \infty$$

must hold, the BC lemmas give

$$\mathbb{P}(\limsup_n E_n) = \begin{cases} 0 & \sum \mathbb{P}(E_n) < \infty \\ 1 & \sum \mathbb{P}(E_n) = \infty \end{cases}.$$

This is a special case of Kolmogorov's 0-1 law!

Def$^{\underline{n}}$ Let $X_1, X_2, \dots$ be a sequence of random variables. Set $\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \dots)$ and $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$. We say $\mathcal{T}$ is a tail σ-algebra.

Some "typical" events in the tail σ-algebra (tail events) are

$$\{\lim X_n \text{ exists}\}, \quad \left\{\sum_{n=1}^{\infty} X_n \text{ converges}\right\}$$

Theorem (Kolmogorov 0-1 law)
Let $X_1, X_2, \dots$ be independent random variables. Then, for every $T \in \mathcal{T}$, either $\mathbb{P}(T) = 0$ or 1. In particular, if ξ is a $\mathcal{T}$ measurable random variable, then $\mathbb{P}(\xi = c) = 1$ for some $c \in [-\infty, \infty]$.

Proof: 1) Define $\mathcal{X}_n = \sigma(X_1, \dots, X_n)$.
Note that $\mathcal{X}_n$ and $\mathcal{T}_n = \sigma(X_{n+1}, \dots)$ are independent for all $n \in \mathbb{N}$.

2) Since $\mathcal{T} \subseteq \mathcal{T}_n$ for all n, $\mathcal{T}$ and $\mathcal{X}_n$ are independent.

3) $\mathcal{X}_\infty = \sigma(X_1, X_2, \ldots)$ and $\mathcal{T}$ are independent because $\bigcup_{n\geq 1} \mathcal{X}_n$ is a π system that generates $\mathcal{X}_\infty$

4) $\mathcal{T} \subseteq \mathcal{X}_\infty$ and $\mathcal{T}$ is independent of itself !!

So for any $F \in \mathcal{T}$, $\underset{\mathbb{P}(F)}{\underset{\shortparallel}{\mathbb{P}(F \cap F)}} = \mathbb{P}(F)\mathbb{P}(F)$.

Hence $P(F) = x$ is a solution to $x = x^2$

$\Rightarrow x \in \{0, 1\}$.

Now if ξ is $\mathcal{T}$ measurable we set

$$c = \sup\{x : P(\xi \leq x) = 0\} \in [-\infty, \infty].$$

Then, $\{\xi < c\} = \bigcup_n \underbrace{\{\xi \leq c - \tfrac{1}{n}\}}_{\text{null set by def}} \Rightarrow \mathbb{P}(\{\xi < c\}) = 0$

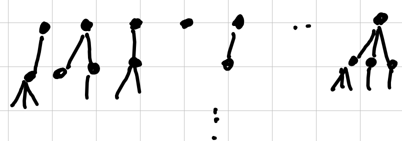
and

$\{\xi \leq c\} = \bigcap_n \underbrace{\{\xi \leq c + \tfrac{1}{n}\}}_{\substack{\text{not null by def} \\ \text{so by 0-1 law almost sure}}} \Rightarrow \mathbb{P}(\{\xi \leq c\}) = 1$

Hence $\mathbb{P}(\{\xi = c\}) = 1$.

"Application" (Extra / non-examinable)

We consider a population (e.g. of humans) for which every member has offspring independent of other members & exactly one ancestor (e.g. matrileneal or patrilineal descent):



generation n
— " — $n+1$

This is called a Galton-Watson process.

The following is a tail event: Let v_n be an arbitrary member of the population at generation n. Then

$\{$a randomly picked individual in a population will eventually be descended from v_n $\} \in \mathcal{T}$.

Hence by the Kolmogorov 0-1 law, eventually everyone or noone will be descended from v_n.

NB: Recent studies into Y chromosomes & mitochondrial DNA suggest that the last time this happened was 280,000 years ago (patrilineal) and 160,000 years ago (matrilineal).